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# TOPOLOGICAL PROPERTIES OF PRODUCTS OF ORDINAL NUMBERS (Set theory of the reals)

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## TOPOLOGICAL PROPERTIES OF PRODUCTS OF ORDINAL NUMBERS

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### 1. INTRODUCTION

The greek letters  $\alpha, \beta, \gamma, \dots$  denote ordinal numbers with the usual order topologies. A space means a  $T_1$  (every one point set is closed) topological space. A space  $X$  is *regular* if every point  $x \in X$  and every closed set  $F$  with  $x \notin F$  are separated by disjoint open sets. It is easy to verify that:

- (1) If  $X$  and  $Y$  are regular, then so is the product space  $X \times Y$ .
- (2) If  $X$  is regular and  $Y \subset X$ , then  $Y$  is regular.

A space  $X$  is *normal* if every pair of disjoint closed sets are separated by disjoint open sets. All compact spaces are normal and all subspaces of ordinal numbers are also normal. On the other hand,  $X = (\omega_1 + 1) \times \omega_1$  is not normal. Indeed, using the Pressing Down Lemma, we can show that the diagonal  $\{(\alpha, \alpha) \in X : \alpha < \omega_1\}$  and the set  $\{\omega_1\} \times \omega_1$  cannot be separated by disjoint open sets. Therefore:

- (1)  $X = (\omega_1 + 1)$  and  $Y = \omega_1$  are normal but  $X \times Y$  is not normal.
- (2)  $X = (\omega_1 + 1)^2$  is compact so normal, but the subspace  $Y = (\omega_1 + 1) \times \omega_1$  of  $X$  is not normal.

Thus the notion of normality is completely different from that of regularity.

The simplest non-trivial space is  $\omega + 1$ , that is, the convergent sequence with its unique limit point. The following famous result was proved by Dowker [Do]:

**Dowker's Theorem.** *If  $X$  is normal, then  $X \times (\omega + 1)$  is normal iff  $X$  is countably paracompact.*

Here a space  $X$  is said to be *countably paracompact* (*countably metacompact*) if for every countable open cover  $\mathcal{U} = \{U_n : n \in \omega\}$ , there is a locally finite (point finite, respectively) open refinement  $\mathcal{V}$  of  $\mathcal{U}$ , where  $\mathcal{V}$  is locally finite (point finite) if for every  $x \in X$ , there is a neighborhood  $U$  of  $x$  such that  $\{V \in \mathcal{V} : V \cap U \neq \emptyset\}$  is finite (if for every  $x \in X$ ,  $\{V \in \mathcal{V} : x \in V\}$  is finite, respectively), moreover an open cover  $\mathcal{V}$  is said to be an open refinement of  $\mathcal{U}$  if for every  $V \in \mathcal{V}$ , there is  $U \in \mathcal{U}$  with  $V \subset U$ .

Dowker asked in [Do] whether there exists a normal space which is not countably paracompact. More than twenty years later, M. E. Rudin constructed in [Ru] such a space in ZFC.

In these connections, we present more definitions. A space  $X$  is *CollectionWise Normal* (abbreviated as CWN) if for every discrete collection  $\mathcal{F}$  of closed sets of  $X$ , there exists a disjoint (equivalently, discrete) collection  $\mathcal{U} = \{U(F) : F \in \mathcal{F}\}$

of open sets with  $F \subset U(F)$ , where a collection  $\mathcal{F}$  is discrete if for every  $x \in X$ , there is a neighborhood  $U$  of  $x$  with  $|\{F \in \mathcal{F} : U \cap F \neq \emptyset\}| \leq 1$ . A space  $X$  is *expandable* if for every locally finite collection  $\mathcal{F}$  of closed sets, there is a locally finite collection  $\mathcal{U} = \{U(F) : F \in \mathcal{F}\}$  of open sets with  $F \subset U(F)$ . For every space, it is not difficult to verify:

- (1) CWN  $\rightarrow$  normal.
- (2) expandable  $\rightarrow$  countably paracompact  $\rightarrow$  countably metacompact.
- (3) normal + countably metacompact  $\rightarrow$  countably paracompact.
- (4) normal + expandable  $\leftrightarrow$  CWN + countably paracompact.
- (5)  $(\omega_1 + 1) \times \omega_1$  is expandable but not normal.

## 2. RESULTS

In the past 10 years, we have investigated such topological properties described in section 1 of product spaces of ordinal numbers. In this section,  $\alpha$  denotes an arbitrary large ordinal number. First we proved in [KOT]:

**Theorem 1.** *For every pair of subspaces  $A$  and  $B$  of  $\alpha$ ,*

- (1)  $A \times B$  is normal iff it is CWN.
- (2)  $A \times B$  is countably paracompact iff it is expandable.
- (3) If  $A \times B$  is normal, then it is countably paracompact. Note that  $(\omega_1 + 1) \times \omega_1$  is countably paracompact but not normal.
- (4) For every pair of subspaces  $A$  and  $B$  of  $\omega_1$ ,  $A \times B$  is normal iff it is countably paracompact iff  $A$  or  $B$  are non-stationary or  $A \cap B$  is stationary. Thus, if  $A$  and  $B$  are disjoint stationary sets of  $\omega_1$ , then  $A \times B$  is neither normal nor countably paracompact.

We asked in [KOT]:

- (a) Is  $A \times B$  countably metacompact for every pair of subspaces  $A$  and  $B$  of  $\alpha$ ?
- (b) Are normality and CWN equivalent for *all* subspaces of  $\alpha^2$ ?
- (c) Are countable paracompactness and expandability equivalent for *all* subspaces of  $\alpha^2$ ?

On (a), we got in [KS1] and [KS2]:

**Theorem 2.**

- (1) All subspaces of  $\alpha^2$  are countably metacompact.
- (2) All subspaces of  $\omega_1^n$  are countably metacompact for every  $n \in \omega$ .
- (3) There is a subspace of  $\omega_1^\omega$  which is not countably metacompact.

After then we got an affirmative answer of (b) in [KNSY]:

**Theorem 3.** *Normality and CWN are equivalent for all subspaces of  $\alpha^2$ .*

However, the question (c) still remains open.

In connection with (4) of Theorem 1, we asked in [KNSY]:

- (d) Are normality and countable paracompactness equivalent for all subspaces of  $\omega_1^2$ ?

On (d), we proved in [KSS]:

**Theorem 4.** *For every subspace  $X$  of  $\omega_1^2$ ,*

- (1)  *$X$  is normal iff  $X$  is expandable iff  $X$  is countably paracompact and strongly collectionwise Hausdorff, where a space is strongly collectionwise Hausdorff (collectionwise Hausdorff) if for every subset  $F$  of  $X$  with the collection  $\{\{x\} : x \in F\}$  discrete, there is a discrete (disjoint, respectively) collection  $\mathcal{U} = \{U(x) : x \in F\}$  of open sets with  $x \in U(x)$ .*
- (2) *If  $V=L$  or the Product Measure Extension Axiom are assumed, then  $X$  is normal iff  $X$  is countably paracompact.*
- (3)  *$X$  is collectionwise Hausdorff.*

This theorem also says that the question (c) is closely related to (d).

### 3. ON THE QUESTION (d)

Now we conjecture that there is a model in which (d) is not true, that is, there is a countably paracompact but not normal subspace of  $\omega_1^2$ . American young mathematicians Eisworth, Just, Pavlov, Smith, Szeptycki are working on this problem. In discussion with them, we have had a candidate of such a subspace. The remaining is an unpublished work with them.

Let  $\text{Lim} = \{\alpha < \omega_1 : \alpha \text{ is limit}\}$  and  $\text{Succ} = \omega_1 \setminus \text{Lim}$ . For each  $\alpha \in \text{Lim}$ , fix a strictly increasing  $\omega$ -sequence  $L_\alpha$  cofinal in  $\alpha$ , moreover for simplicity of our discussion we assume  $L_\alpha \subset \text{Succ}$ . Then we call  $\mathcal{L} = \{L_\alpha : \alpha \in \text{Lim}\}$  a ladder system. Set  $L(\mathcal{L}) = \bigcup_{\alpha \in \text{Lim}} L_\alpha$ , then  $L(\mathcal{L}) \subset \text{Succ}$ . The ladder space  $X(\mathcal{L})$  determined by  $\mathcal{L}$  is defined as follows:

$$X(\mathcal{L}) = \left[ \bigcup_{\alpha \in L(\mathcal{L})} \{\alpha\} \times \{\beta \in \text{Lim} : \alpha < \beta\} \right] \cup \left[ \bigcup_{\alpha \in \text{Lim}} (\{\alpha\} \cup L_\alpha) \times \{\alpha + 1\} \right].$$

This is our candidate. The following are proved in our discussion:

- (1) In ZFC,  $X(\mathcal{L})$  is not normal for every ladder system  $\mathcal{L}$ .
- (2) If  $\text{MA}(\omega_1)$  is assumed, then for every ladder system  $\mathcal{L}$ ,  $X(\mathcal{L})$  is not countably paracompact. In fact,  $\text{MA}(\omega_1)$  destroys the property (+) below.
- (3) In ZFC,  $X(\mathcal{L})$  is not countably paracompact for some ladder system  $\mathcal{L}$ .

So our conjecture is:

- (d') In some model, there is a ladder system  $\mathcal{L}$  such that  $X(\mathcal{L})$  is countably paracompact.

Finally we present a combinatorial equivalent property due to Pavlov and Szeptycki, independently.

- (4) Let  $\mathcal{L}$  be a ladder system. Then  $X(\mathcal{L})$  is countably paracompact iff  $\mathcal{L}$  satisfies the following two properties (WU) and (+):

- (WU)  $\forall f : \text{Lim} \rightarrow \omega \exists g : L(\mathcal{L}) \rightarrow [\omega]^{<\omega} \forall \alpha \in \text{Lim} (|\{\beta \in L_\alpha : f(\alpha) \notin g(\beta)\}| < \omega)$ .
- (+)  $\forall f : L(\mathcal{L}) \rightarrow \omega \{\alpha \in \text{Lim} : |f''L_\alpha| = \omega\}$  is not stationary.

So the conjecture (d') can be written as:

- (d'') In some model, there is a ladder system  $\mathcal{L}$  satisfying both (WU) and (+).

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